Open and Closed Sets

Definition Let X be a set and $\mathscr{T} = \{U \mid U \subseteq X\}$ be a collection of subsets of X. Then $\mathscr{T}$ is called a topology on X if

(1) $\emptyset$ and X are in $\mathscr{T}$.

(2) The union of the elements of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

A set X for which a topology $\mathscr{T}$ has been specified is called a topological space.

Properly speaking, a topological space is an ordered pair $(X, \mathscr{T})$ consisting of a set X and a topology $\mathscr{T}$ on X, but we often omit specific mention of $\mathscr{T}$ if no confusion will arise.

Definition Let X be a topological space with topology $\mathscr{T}$. A subset U of X is called an open set of X if $U \in \mathscr{T}$, i.e. U belongs to the collection $\mathscr{T}$.

A subset F of X is called an closed set of X if $F^c = X \setminus F \in \mathscr{T}$, i.e. the complement subset of F in X is an open set of X.

A subset N is called a neighborhood of p if we can find an open set U such that

$$p \in U \subseteq N.$$

Note that if N is an open subset of X, then it is a neighborhood of each point $p \in N$.

Remark Let X be a topological space. Then the following conditions hold:

(1) $\emptyset$ and X are open.

(2) Arbitrary union of open sets is open.

(3) Any finite intersection of open sets is open.

Examples

1. Let $X = \mathbb{E}^n$ and let $\mathscr{T} = \{U \mid \forall x \in U,\ \exists \varepsilon = \varepsilon(x) > 0 \text{ s.t. } B_\varepsilon(x) \subset U\}$, where $B_\varepsilon(x) = \{y \in \mathbb{E}^n \mid d(x,y) < \varepsilon\}$ denotes the Euclidean ball with center x and radius ε. Then $\mathscr{T}$ is a topology on $\mathbb{E}^n$ and it is called the usual or standard topology on $\mathbb{E}^n$.
2. Let X be a set and let $\mathscr{T} = \mathscr{P}(X) = \{U \mid U \subset X\}$ be the collection of all subsets of X, called the power set of X. Then $\mathscr{T}$ is a topology on X and it is called the discrete topology on X.
3. Let X be a set and let $\mathscr{T} = \{\emptyset, X\}$ consist of $\emptyset$ and X only. Then $\mathscr{T}$ is a topology on X and it is called the trivial topology on X.
4. Let $X = \mathbb{R}$ and let

$$\mathscr{T}_f = \{U \mid X \setminus U \text{ is either finite subset or all of } X\}.$$

Then $\mathscr{T}_f$ is a topology on X, called the the finite complement topology.

Remark Let X be a topological space. Then the following conditions hold:

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersection of closed sets is closed.

(3) Any finite union of closed sets is closed.

Definition Let X be a topological space with topology $\mathscr{T}$ and let Y be a subset of X. Then the collection

$$\mathscr{T}_Y = \{Y \cap U \mid U \in \mathscr{T}\}$$

is a topology on Y, called the subspace or induced topology. With this topology, Y is called a subspace of X; its open sets consists of all intersections of open sets of X with Y.

Example Let $X = \mathbb{R}$ be the real line with the usual topology $\mathscr{T} = \{(a, b) \mid a < b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and let $Y = [0, 1]$. Then the subspace topology $\mathscr{T}_Y$ consists a set of the following types:

$$(a, b) \cap Y = \begin{cases} (a, b) & \text{if } a \text{ and } b \text{ are in } Y \\ [0, b) & \text{if only } b \text{ is in } Y \\ (a, 1] & \text{if only } a \text{ is in } Y \\ Y \text{ or } \emptyset & \text{if neither } a \text{ nor } b \text{ is in } Y \end{cases}$$

Example Let $X = \mathbb{R}$ be the real line with the usual topology $\mathscr{T} = \{(a, b) \mid a < b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and let $Y = [0, 1) \cup \{2\}$. Then the one-point set $\{2\}$ is open in the subspace topology $\mathscr{T}_Y$.

Definition Let A be a subset of a topological space X. A point p of X is called a limit point (or accumulation point) of A if every neighborhood N of p contains at least one point of $A \setminus \{p\}$, i.e.

$$N \cap A \setminus \{p\} \neq \emptyset.$$

Let A' denote the set of limit points of A. Note that a limit point of A may not be a point in A.

Examples

1. Let $X = \mathbb{R}$ and let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then A has exactly one limit point, namely the origin.
2. Let $X = \mathbb{R}$ and let $A = [0, 1)$. Then $[0, 1]$ is the set of limit points of A.
3. Let $X = \mathbb{E}^3$ and let $A = \{(x, y, z) \mid x, y, z \in \mathbb{Q}\}$. Then $\mathbb{E}^3$ is the set of limit points of A.
4. Let $X = \mathbb{E}^3$ and let $A = \{(x, y, z) \mid x, y, z \in \mathbb{Z}\}$. Then A does not have any limit points.
5. Let $X = \mathbb{R}$ with the finite complement topology $\mathscr{T}_f$. If we take A to be an infinite subset of X, then every point of X is a limit point of A. On the other hand a finite subset of X has no limit points in this topology.

Theorem A set is closed if and only if it contains all of its limit points.

Proof If A is closed, then $X \setminus A$ is open. Since

$$A \cap (X \setminus A) \setminus \{p\} = \emptyset \quad \forall\, p \in X \setminus A,$$

$X \setminus A$ does not contain any limit point of A.

Therefore A contains all of its limit points.

Conversely, suppose A contains all of its limit points and let $p \in X \setminus A$. Since p is not a limit point of A, there is a neighborhood N of p such that

$$N \cap A = \emptyset \implies p \in N \subset X \setminus A$$

This implies that $X \setminus A$ is a neighborhood of each of its points and consequently open. Therefore A is closed.

Definition The union of A and all its limit points is called the closure of A and is written $\bar{A} = A \cup A'$.

Theorem The closure of A is the smallest closed set containing A, in other words the intersection of all closed sets which contain A.

Proof For each $p \in X \setminus \bar{A}$, there exists an open neighborhood U of p such that

$$U \cap A = \emptyset.$$

Since U is an open neighborhood of each of its points, U cannot contain any of the limit points of A either, i.e.

$$U \cap A' = \emptyset \implies U \cap \bar{A} = \emptyset \implies p \in U \subseteq X \setminus \bar{A} \implies X \setminus \bar{A} \text{ is open and } \bar{A} \text{ is closed}$$

Let B be a closed set which contains A. Then

$$B' \subseteq B \implies \bar{B} = B \cup B' \subseteq B \subseteq \bar{B} \implies B = \bar{B}$$

and

$$A' \subseteq B' \implies \bar{A} = A \cup A' \subseteq B \cup B' = \bar{B} = B \implies \bar{A} \subseteq B \implies \bar{A} \subseteq \bigcap_{A \subseteq B \text{ is closed}} B$$

Since $\bar{A}$ is closed and $A \subseteq \bar{A}$, this implies that $\bar{A} \in \{B \mid B \text{ is closed containing } A\}$ and thus

$$\bigcap_{A \subseteq B \text{ is closed}} B \subseteq \bar{A} \implies \bar{A} = \bigcap_{A \subseteq B \text{ is closed}} B.$$

Corollary A set is closed if and only if it is equal to its closure.

Definitions Let A be a subset of a topological space X. The interior of A, usually denoted $\overset{o}{A}$, is the union of all open sets contained in A, i.e.

$$\overset{o}{A} = \bigcup_{U \text{is open and } U \subseteq A} U = \text{ the largest open set contained in } A$$

The frontier (or boundary) of A, usually denoted ∂A, is the intersection of the closure of A with the closure of $X \setminus A$, i.e.

$$\partial A = \bar{A} \cap \overline{X \setminus A}$$

Definition If X is a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is at least one basis element B containing $x \implies \bigcup_{B \in \mathscr{B}} B = X.$

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that

$$B_3 \subseteq B_1 \cap B_2$$

i.e. For any B_1, $B_1 \in \mathscr{B}$ satisfy $B_1 \cap B_2 \neq \emptyset$, there is a $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Definition If $\mathscr{B}$ is a basis for a topology on X, the topology $\mathscr{T}$ generated by $\mathscr{B}$ is described as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathscr{T}$) if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$, i.e.

$$\mathscr{T} = \{U \mid \forall\, x \in U,\, \exists\, B \in \mathscr{B} \text{ s.t. } x \in B \text{ and } B \subset U\}$$

Remark Note that each element of $\mathscr{B}$ is open in X under this definition, so that $\mathscr{B} \subset \mathscr{T}$.

It is easy to check that the collection $\mathscr{T}$ generated by the basis $\mathscr{B}$ is, in fact, a topology on X.

- $\emptyset \in \mathscr{T}$ since it satisfies the defining condition of openess vacuously.
- $X \in \mathscr{T}$ since for each $x \in X$ there is a basis element containing x and contained in X.
- Let $\{U_\alpha\}_{\alpha \in J}$ be a collection of elements of $\mathscr{T}$ and let $U = \bigcup_{\alpha \in J} U_\alpha$. Then $U \in \mathscr{T}$ since for each $x \in U$, there is an index α such that $x \in U_\alpha$ and since $U_\alpha \in \mathscr{T}$ there is a basis element B such that
$$x \in B \subseteq U_\alpha \implies x \in B \text{ and } B \subseteq U \implies U \in \mathscr{T}$$
- Let $\{U_i\}_{1 \le i \le n}$ be a finite collection of elements of $\mathscr{T}$ and let $U = \bigcap_{i=1}^{n} U_i$. Then $U \in \mathscr{T}$.

Theorem Let β be a nonempty collection of subsets of a set X. If the intersection of any finite members of β is always in β, and if $\cup\beta = X$, then β is a basis for a topology on X.

Proof Take the obvious candidate, namely the collection of all unions of members of β as the open sets, then check the requirements for a topology.

Continuous Functions

Definition Let X and Y be topological spaces. A function $f : X \to Y$ is continuous on X if for each point x of X and each neighborhood N of $f(x)$ in Y the set $f^{-1}(N)$ is a neighborhood of x in X.

A continuous function is very often called a map for short (in this book).

A function $h : X \to Y$ is called a homeomorphism if it is one-to-one, onto, continuous and has a continuous inverse $h^{-1} : Y \to X$. When such a function exists, X and Y are called homeomorphic (or topologically equivalent) spaces.

Theorem Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if the inverse image of each open set of Y is open in X.

Proof

($\Rightarrow$) If $f : X \to Y$ is continuous and if O is an open subset of Y, then O is a neighborhood of each of its points and, by the definition of continuity, $f^{-1}(O)$ must be a neighborhood of each of its points in X. Hence $f^{-1}(O)$ is an open set in X.

($\Leftarrow$) For each point x of X and each neighborhood N of $f(x)$ in Y, there is an open subset O in Y such that
$$f(x) \in O \subseteq N \quad \text{and } f^{-1}(O) \text{ is open in } X.$$

Since
$$x \in f^{-1}(O) \subseteq f^{-1}(N),$$

$f^{-1}(N)$ is a neighborhood of x, f is continuous at x. Since f is continuous for each $x \in X$, f is continuous on X.

Example Let $X = [0,1) \subset \mathbb{R}$, $Y = C \subset \mathbb{C}$ be the unit circle in the complex plane $\mathbb{C}$ and let $f : [0,1) \to C$ be defined by

$$f(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x \quad \text{for each } x \in [0,1).$$

Note that f is continuous, one-to-one, onto, but its inverse $f^{-1} : C \to [0,1)$ is not continuous (e.g. $\left(f^{-1}\right)^{-1}([0,1/2)) = \{p \in C \mid f^{-1}(p) \in [0,1)\}$ is not open in C while $[0,1/2)$ is open in $[0,1)$).

Theorem Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then the composition $g \circ f : X \to Z$ is a continuous function.

Proof Let O be an open set in Z. Since

$$(g \circ f)^{-1}(O) = f^{-1} g^{-1}(O)$$

and $g^{-1}(O)$ is open in Y because g is continuous, so $f^{-1} g^{-1}(O)$ must be open in X by the continuity of f. Therefore $g \circ f : X \to Z$ is continuous.

Theorem Suppose $f : X \to Y$ is continuous, and let $A \subseteq X$ have the subspace topology. Then the restriction $f\,|_A : A \to Y$ is continuous.

Proof Let O be an open set in Y and notice that

$$\left(f\,|_A\right)^{-1}(O) = A \cap f^{-1}(O).$$

Since f is continuous, $f^{-1}(O)$ is open in X. Therefore $\left(f\,|_A\right)^{-1}(O)$ is open in the subspace topology on A, and the continuity of $f\,|_A$ follows from the preceding Theorem.

Definition The map $1_X : X \to X$, defined by $1_X(x) = x$ for each $x \in X$, is called the identity map of X. If we restrict 1_X to a subspace A of X we obtain the inclusion map $i : A \to X$.

Theorem The following are equivalent:

(a) $f : X \to Y$ is continuous.

(b) If β is a base for the topology of Y, the inverse image of every member of β is open in X.

(c) $f(\bar{A}) \subseteq \overline{f(A)}$ for any subset A of X.

(d) $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for any subset B of Y.

(e) The inverse image of each closed set in Y is closed in X.

Proof

$[(a) \Rightarrow (b)]$ For each $B \in \beta$, since B is an open set in the topology generated by β, $f^{-1}(B)$ is open in X.

$[(b) \Rightarrow (c)]$ Let A be a subset of X. Since $\bar{A} = A \cup A'$ and $f(A) \subseteq \overline{f(A)} = f(A) \cup f(A)'$, it suffices to show that if $x \in \bar{A} \setminus A$ and if $f(x) \notin f(A)$, then $f(x) \in f(A)'$.

Suppose that $x \in \bar{A} \setminus A$, $f(x) \notin f(A)$ and N is a neighborhood of $f(x)$ in Y. Since β is a base for the topology of Y, there exists a basis element (an open subset) B in β such that

$$f(x) \in B \subseteq N \implies x \in f^{-1}(B) \subseteq f^{-1}(N).$$

Assuming (b), the set $f^{-1}(B)$ is open in X and is therefore a neighborhood of x. Also since

$$x \in A' \implies f^{-1}(B) \cap A \neq \emptyset \implies B \cap f(A) \neq \emptyset$$

and since

$$B \cap f(A) \subseteq N \cap f(A) \implies N \cap f(A) \setminus \{f(x)\} = N \cap f(A) \neq \emptyset \implies f(x) \in f(A)'$$

This completes the proof of (c).

$[(c) \Rightarrow (d)]$ For any subset B of Y, since $f^{-1}(B)$ is a subset of X and by assuming (c), we have

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f(f^{-1}(B))} = \bar{B} \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$$

$[(d) \Rightarrow (e)]$ If B is a closed subset of Y, since $\bar{B} = B$ and by assuming (d), we have

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B}) = f^{-1}(B) \subseteq \overline{f^{-1}(B)} \implies f^{-1}(B) = \overline{f^{-1}(B)}$$

and thus $f^{-1}(B)$ is closed in X.

$[(e) \Rightarrow (a)]$ For each open set O of Y, since

$$X \setminus f^{-1}(O) = \{x \in X \mid f(x) \notin O\} = \{x \in X \mid f(x) \in Y \setminus O\} = f^{-1}(Y \setminus O),$$

$Y \setminus O$ is closed in Y and by assuming (e), we have $f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$ is closed in X and thus $f^{-1}(O)$ is open in X. This shows that $f : X \to Y$ is continuous.